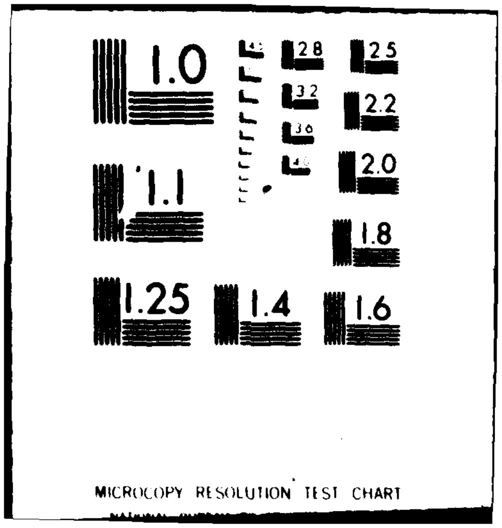
AD-A115 392 STANFORD UNIV CA DEPT OF STATISTICS F/G 12/1
RANDOM COVERING AND PACKING ON THE LINE.(U)
FEB 82 H J WEINER N00014-76-C-0475
UNCLASSIFIED TR-314 NL

1 of 1
AD A
115392

END
DATE
FILMED
7-82
DTIC



MICROCOPY RESOLUTION TEST CHART

RANDOM COVERING AND PACKING ON THE LINE

By

HOWARD J. WEINER

TECHNICAL REPORT NO. 314

February 16, 1982

Prepared Under Contract
N00014-76-C-0475 (NR-042-267)
For the Office of Naval Research

Herbert Solomon, Project Director

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

Random Covering and Packing on the Line

by Howard J. Weiner

University of California, Davis and Stanford University

I. Introduction: Covering Model. Unit length segments are sequentially placed independently at random uniformly on a line segment $[0,x)$ until that segment is completely covered. The covering models of Renyi and Solomon will be considered in section 2, and asymptotic results for the two dimensional cases for both models will be given in section 3. The variance and other extensions will be indicated.

A packing and covering model for the one dimensional abacus grid will be indicated. The one-dimensional Renyi model is as follows. The center of each unit segment is chosen uniformly on $[0,x)$. The unit segment so chosen is placed on $[0,x)$, $x > 1$. The process is repeated, and a new unit segment is kept (parked) if and only if its center is on an uncovered portion of $[0,x)$, otherwise it is discarded. The process stops as soon as the $[0,x)$ is completely covered. All unit segments except those covering the ends must lie within $[0,x)$.

Let $M(x)$ = mean of the minimum number of unit segments covering $[0,x)$ in the Renyi model. (1.1)

The Solomon model allows the center of each unit segment to be chosen uniformly on $[0,x)$. Unit segments are placed without exclusion or further constraint until the $[0,x)$ is covered. Let

$R(x)$ = mean of the minimum number of unit segments covering $[0,x)$ in the Solomon model. (1.2)

Results for a model related to the Solomon model (1.2) are obtained in Solomon (1966). The results here are by different considerations.

The parking models (Blaisdell et al (1970), Solomon (1966) are related to these problems and references relevant to sections I, II, VII are contained there.

let, in either model,

$X_1(t)$, $X_2(t)$ be I.I.D., each distributed as $X(t)$, which is the minimum total number of unit length cars needed to cover $[0,t)$. (1.3)

Then conditional on the first placement with center at $t + \frac{1}{2}$, one may represent $X(x)$ as

$$X(x) = X_1(t) + X_2(x-t-1) + 1 \tag{1.4}$$

Taking expectations of (1.4) for the Renyi model and combining two integrals on the right, one obtains

$$M(x) = 1 + \frac{2}{x}\int_0^{x-\frac{1}{2}} M(u)du, \; x \geq \frac{1}{2} \tag{1.5}$$
$$\text{and } M(x) = 1, \; 0 < x \leq \tfrac{1}{2}.$$

Hence for $x \geq 1$,

$$M(x) = 1 + \frac{1}{x} + \frac{2}{x}\int_{\frac{1}{2}}^{x-\frac{1}{2}} M(u)du. \tag{1.6}$$

Lemma 1. If $\Delta(x)$, $x \geq 1$, satisfies

$$\Delta(x) \underset{(\leq)}{\leq} \frac{2}{x}\int_{\frac{1}{2}}^{x-\frac{1}{2}} \Delta(u)du \tag{1.7}$$

and

$$\Delta(x) \underset{(\leq)}{\geq} 0, \text{ for} \tag{1.8}$$
$$1 \leq x_0 \leq x \leq x_0 + \tfrac{1}{2},$$

then

$$\Delta(x) \underset{(\leq)}{\geq} 0 \quad \text{for all } x \geq x_0. \tag{1.9}$$

Proof. Assume the upper inequality in (1.7) and (1.8). Then (1.8) into (1.7) for $x_0 + \frac{1}{2} \leq x \leq x_0 + 1$ yields

$$\Delta(x) = \frac{2}{x}\left[\int_{\frac{1}{2}}^{x_0} \Delta(u)du + \int_{x_0}^{x-\frac{1}{2}} \Delta(u)du\right] \tag{1.10}$$

The first term on the right of (1.10) is non-negative since $\Delta(x_0 + \frac{1}{2}) \geq 0$ and the integrand of the second term is non-negative by (1.8). The result now holds for $x_0 \leq x \leq x_0 + 1$, so that the process is continued as before, and an induction completes the proof. The lower inequality is similar.

Theorem 1. For $x \geq 1$,

$$1.485\, x + .485 \leq M(x) \leq 1.5x + .5 \tag{1.11}$$

Proof. An explicit computation verifies that if

$$L_1(x) \equiv 1.485x + .485 \tag{1.12}$$

$$L_2(x) \equiv 1.5x + .5$$

then for $x \geq 1$,

$$L_1(x) \leq 1 + \frac{1}{x} + \frac{2}{x}\int_{\frac{1}{2}}^{x-\frac{1}{2}} L_1(u)du \qquad \text{(1.13)(i)}$$

$$L_2(x) = 1 + \frac{1}{x} + \frac{2}{x}\int_{\frac{1}{2}}^{x-\frac{1}{2}} L_2(u)du \qquad \text{(1.13)(ii)}$$

and for $1 \leq x \leq \frac{3}{2}$,

$$L_1(x) \leq M(x) \leq L_2(x). \tag{1.14}$$

Hence, defining, for $i=1,2$,

$$\Delta_i(x) = L_i(x) - M(x), \tag{1.15}$$

it follows that

$$\Delta_1(x) \leq \frac{2}{x}\int_{\frac{1}{2}}^{x-\frac{1}{2}} \Delta_1(n)du \qquad \text{(1.16)(i)}$$

$$\Delta_2(u) = \frac{2}{x}\int_{\frac{1}{2}}^{x-\frac{1}{2}} \Delta_2(u)du \qquad \text{(1.16)(ii)}$$

and for $1 \leq x \leq \frac{3}{2}$, it is easily computed that

$$\Delta_1(x) \leq 0 \qquad (1.17)$$
$$\Delta_2(x) \geq 0.$$

The theorem now follows from (1.14) - (1.17) and the lemma.

Taking expectations of (1.4) for the Solomon model, one obtains for $x > 0$,

$$R(x) = 1 + \frac{2}{1+x}\int_0^x R(u)du \qquad (1.18)$$

with $R(0) = 1$.

This Volterra-type equation has a unique solution, which is easily seen to be, for $x > 0$

$$R(x) \equiv 2x + 1. \qquad (1.19)$$

To obtain precise asymptotic behavior of $M(x)$, define the Laplace transform for $0 < s < 1$

$$L(s) \equiv \int_{\frac{1}{2}}^{\infty} e^{-sx}M(x)dx. \qquad (1.20)$$

Since $M(x) < 2x + 1$, $L(s)$ is well defined. From (1.5) it immediately follows that

$$L'(s) + \frac{2e^{-s/2}}{s} L(s) = \frac{2e^{-s}}{s^2} - e^{-s/2}\left(\frac{1}{2s} + \frac{3}{s^2}\right) \qquad (1.21)$$

with solution, using the fact that $M(x) \leq cx$,

$$L(s) = e^{2\int_s^{\infty}\frac{e^{-u/2}}{u}du}\int_s^{\infty} e^{-2\int_v^{\infty}\frac{e^{-u/2}}{u}du}\left[e^{-v/2}\left(\frac{1}{2v} + \frac{3}{v^2}\right) - \frac{2e^{-v}}{v^2}\right]dv \qquad (1.22)$$

Since for $0 < s < 1$,

$$\int_s^\infty \frac{e^{-u/2}}{u}\,du = -\gamma - \int_0^{s/2} \frac{e^{-u}-1}{u}\,du - \ln(\tfrac{s}{2}), \tag{1.23}$$

for $0 < s \ll 1$, it is seen by a Taylor expansion to one term,

$$L(s) \simeq \frac{c}{s^2} + \frac{c-1}{s} \tag{1.24}$$

where

$$c = 4e^{-2\gamma}\int_0^\infty e^{-2\int_v^\infty \frac{e^{-u/2}}{u}du}\left[e^{-v/2}\left(\frac{1}{2v} + \frac{3}{v^2}\right) - \frac{2e^{-v}}{v^2}\right]dv. \tag{1.25}$$

<u>Theorem 2</u>.

$$\lim_{x\to\infty} x^{-1}M(x) = c, \tag{1.26}$$

$$\lim_{x\to\infty} x^{-1}R(x) = 2, \tag{1.27}$$

and

$$1.485 < c < 1.50 \tag{1.28}$$

Since it may be easily shown that $M'(x) > 0$ from (1.5), Abelian and Tauberian theorems (Widder (1941), pp. 182, 192) yield from (1.24), (1.25) that (1.26) holds. The equation (1.27) is immediate from (1.19).

To show that $c < 1.5$ in (1.28), taking the Laplace transform of the second equation of (1.13), where $L_2(x) \equiv 1.5x + 1.5$ and repeating steps as in (1.21) - (1.25) one obtains that

$$\lim_{x \to \infty} x^{-1} L_2(x) = \frac{3}{2} = \tag{1.29}$$

$$= 4e^{-2\gamma} \int_0^\infty e^{-2\int_v^\infty \frac{e^{-u/2}}{u} du} \left[\frac{2e^{-v}}{v} + \frac{3e^{-v}}{2v^2} + \frac{3e^{-v/2}}{s^3} (e^{-s/2} - e^{-s}) - \frac{4e^{-3s/2}}{s^2} \right]$$

$$= 4e^{-2\gamma} \int_0^\infty e^{-2\int_v^\infty \frac{e^{-u/2}}{u} du} \left[\frac{2e^{-v}}{v} + \frac{3e^{-v}}{2v^2} + \frac{3e^{-v/2}}{v^3} (e^{-v/2} - e^{-v}) - \frac{4e^{-3v/2}}{v^2} \right] dv.$$

By evaluating the contribution to the integral in (1.29) near 0, and comparing it to the contribution near 0 to the integral in (1.25), one obtains that these are not equal.

To establish the left inequality in (1.28), a similar method to that used above suffices. This is outlined as follows. It is easily established that there exists an α such that

$$\ell(x) \equiv \begin{cases} 1.485x + .485 & , \quad x \geq \frac{1}{2} \\ 1 & , \quad x < \frac{1}{2} \end{cases} \tag{1.30}$$

satisfies, for $x \geq \frac{1}{2}$,

$$\ell(x) = 1 + \frac{\alpha}{x} + \frac{2}{x} \int_{\frac{1}{2}}^{x - \frac{1}{2}} \ell(u)du. \tag{1.31}$$

Taking Laplace transforms of (1.31) for $0 < s < 1$,

$$k(s) \equiv \int_{\frac{1}{2}}^\infty e^{-su} \ell(u)du, \tag{1.32}$$

one obtains an explicit solution for $k(s)$, and then that for $0 < s \ll 1$,

$$k(s) \simeq \frac{\beta}{s^2} + \frac{\beta - 1}{s}, \tag{1.33}$$

where β is expressed as a definite integral which is seen to be not equal to that in (1.25). This completes the proof.

II. Second Moments: Covering Models.

Let

$$M_2(x) = EX^2(x) \tag{2.1}$$
$$R_2(x) = EY^2(x)$$

Theorem II.

$$\lim_{x\to\infty} x^{-1}\mathrm{Var}\ X(x) = \alpha > 0 \tag{2.2}$$

where α is obtained explicitly.

For $x > 0$,

$$\mathrm{Var}\ Y(x) = \frac{2}{3}(x+1) \tag{2.3}$$

Proof. Squaring (1.4) and taking expectations, it follows that

$$M_2(x) = 2M(x) - 1 + \frac{2}{x}\int_0^{x-1} M(t)M(x-1-t)dt \tag{2.4}$$
$$+ \frac{2}{x}\int_0^{x-\frac{1}{2}} M_2(t)dt.$$

$$R_2(x) = 2R(x) - 1 + \frac{2}{x+1}\int_0^{x-1} R(t)R(x-1-t)dt$$
$$+ \frac{2}{x+1}\int_0^{x} R_2(t)dt. \tag{2.5}$$

with respective initial conditions

(i) $M_2(x) = 1,\ 0 < x < \frac{1}{2}$ (2.6)(i)

(ii) $R_2(0) = 1.$ (2.6)(ii)

Since $R(x) = 2x + 1$, (2.5) has immediate solution

$$R_2(x) = 4x^2 + \frac{14}{3}x + \frac{5}{3}, \tag{2.7}$$

and hence

$$\mathrm{Var}\ Y(x) = R_2(x) - R^2(x) = \frac{2}{3}(x+1), \tag{2.8}$$

which is (2.3).

To prove (2.2), define

$$\ell(x) = \begin{cases} 1, & 0 < x < \frac{1}{2}. \\ cx + c - 1, & x > \frac{1}{2}. \end{cases} \tag{2.9}$$

Then, defining, for $x > 1$,

$$D(x) \equiv E(X(x) - \ell(x)^2, \tag{2.10}$$

it is obtained that

$$D(x) = f(x) + \frac{2}{x}\int_0^{x-\frac{1}{2}} (M(t) - \ell(t))(M(x-1-t)-\ell(x-1-t))dt$$

$$+ \frac{2}{x}\int_0^{x-\frac{1}{2}} D(t)dt, \tag{2.11}$$

where

$$\lim_{x\to\infty} x^{-1}(f(x) = 0, \tag{2.12}$$

and $f(x)$ is expressible in terms of $M(x)$.

Hence (2.11) may be written

$$D(x) = g(x) + \frac{2}{x}\int_0^{x-\frac{1}{2}} D(t)dt, \tag{2.13}$$

where, by (1.24), (2.12)

$$\lim_{x \to \infty} x^{-1} g(x) = b \neq 0. \tag{2.14}$$

Solving for the Laplace transform of D(x) as in section I, one may obtain that the asymptotic behavior of D(x) may be established as

$$\lim_{x \to \infty} x^{-1} D(x) = \alpha > 0. \tag{2.15}$$

Noting that

$$|\mathrm{Var}\ X(x) - D(x)| = |M_2(x) - M^2(x) - (M_2(x) - 2\ell(x)M(x) + \ell^2)$$

$$= (M(x) - \ell(x))^2, \tag{2.16}$$

it follows from (1.24) that, as $x \to \infty$, (2.17)

$x^{-1}|\mathrm{Var}\ X(x) - D(x)| = o(x)$, establishing (2.2).

III. Abacus model.

Discrete covering model versions of those in sections I, II are given. A line segment of length $a+1$, with $a > 0$ an integer, is placed on the x-axis so that the left endpoint is 0. For $d > 0$ an integer, a 2d-length "car" is set ("parked") at random upon the segment, such that the midpoint of the "car" is placed upon any of the points (0,1), (0,2), (0,a) uniformly with probability $\frac{1}{a}$. Independent of the first "car," a second 2d-length car is placed with midpoint chosen uniformly at random (if possible) at (0,1), (0,2),...(0,a), and "parked" (kept in that place) if and only if the center of the car is on an uncovered point (0,k),

$1 \leq k \leq a$ of the original segment, otherwise it is discarded, and a new 2d-length car is placed I.I.D. as the previous one. The process continues until for the first time the points $(0,k)$ for $1 \leq k \leq a$ are all covered. This is a Renyi model of covering the one-dimensional abacus. A Solomon model of covering is similar, except that a newly placed d-length car with left endpoint on an integer point is kept ("parked") if and only if some portion of the car covers a previously uncovered integer point $(0,k)$, $1 \leq k \leq a$.

Let, for $a > 0$ an integer, $d > 0$ an integer
$X(a)$ = minimum number of 2d-length cars needed to cover the $[0,a+1]$ interval in the Renyi model (3.1)

Let $Y(a)$ = minimum number of d-length cars needed to cover $[0,a+1]$ in the Solomon model. (3.2)

Denote

$$m(a) \equiv E(X(a)). \tag{3.3)(i}$$

$$r(a) \equiv E(Y(a)). \tag{3.3)(ii}$$

$$\sigma^2(a) \equiv E(X(a)-m(a))^2 \tag{3.3)(iii}$$

$$\tau^2(a) \equiv E(Y(a)-r(a))^2. \tag{3.3)(iv}$$

Theorem III.

$$\lim_{a\to\infty} a^{-1}m(a) = \alpha_1 > 0 \tag{3.4)(i}$$

$$\lim_{a\to\infty} a^{-1}r(a) = \alpha_2 > 0 \tag{3.4)(ii}$$

$$\lim_{a \to \infty} a^{-1}\sigma^2(a) = \beta_1 > 0 \qquad (3.4)(iii)$$

$$\lim_{a \to \infty} a^{-1}\tau^2(a) = \beta_2 > 0. \qquad (3.4)(iv)$$

<u>Proof</u>. Again taking expectations of (1.4), regarding the quantities there as appropriate for either X(a) or Y(a), respectively, yields for the Renyi model,

$$m(a) = 1 + \frac{2}{a}\sum_{\ell=1}^{a-d-1} m(\ell), \qquad (3.5)$$

and $m(1)=m(2)=\ldots,m(d)=m(d+1)=1$.

For the Solomon model,

$$r(a) = 1 + \frac{2}{a+d}\sum_{\ell=1}^{a-1} r(\ell), \qquad (3.6)$$

with $r(1) = 1$.

As in section I, define, for $0 < s < 1$,

$$K(s) = \sum_{\ell=d+2}^{\infty} e^{-s\ell} m(\ell). \qquad (3.7)$$

Since $m(\ell) \leq \ell$, K(s) is well defined.

Clearing of fractions in (3.5) and applying (3.7) yields

$$K'(s) + \frac{2e^{-s(d+1)}}{1-e^{-s}} K(s) = -\left(\frac{e^{-s(d+2)}}{(1-e^{-s})}\right)' - \frac{2(d+1)e^{-s(d+2)}}{1-e^{-s}}$$

$$= \frac{e^{-s(d+2)}}{(1-e^{-s})^2}\,[(d+1)e^{-s}-d] \qquad (3.8)(i)$$

or

$$K'(s) + \frac{2e^{-s(d+1)}}{1-e^{-s}} K(s) = \frac{e^{-s(d+2)}}{(1-e^{-s})^2} [(d+1)e^{-s}-d]. \tag{3.8)(ii}$$

This has solution

$$K(s) = \exp\left[2\int_s^\infty \frac{e^{-u(d+1)}}{1-e^{-u}} du\right] \int_s^\infty \exp\left[-2\int_v^\infty \frac{e^{-u(d+1)}}{1-e^{-u}} du\right] \left[\frac{e^{-v(d+2)}}{(1-e^{-v})^2}\{(d+1)e^{-v}-d\}\right] dv. \tag{3.9}$$

For $0 < s \ll 1$,

$$K(s) \simeq \frac{b}{s^2} + \frac{2(d+1)b-1}{s} , \tag{3.10}$$

where

$$b = \int_0^\infty \exp\left[-2\int_v^\infty \frac{e^{-u(d+1)}}{1-e^{-u}} du + 2\delta\right] \left[\frac{e^{-v(d+2)}}{(1-e^{-v})^2} \{(d+1)e^{-v}-d\}\right] dv, \tag{3.11}$$

with

$$\delta \equiv \int_0^1 \left(\frac{e^{-u(d+1)}}{(1-e^{-u})} - \frac{1}{u} \right) du + \int_1^\infty \frac{e^{-u(d+1)}}{(1-e^{-u})} du \tag{3.12}$$

It is noted that the symbols $\alpha_1 \delta_1 \gamma_1$, etc. are used to denote different quantities in different expressions.

A similar result holds for the Solomon model, by the same method, outlined below.

Let

$$J(s) = \sum_{\ell=2}^{\infty} e^{-s\ell} \, (\ell). \tag{3.13}$$

Then

$$J'(s) = \left(\frac{2e^{-s}}{1-e^{-s}} - d\right) J(s) = \frac{(d+4)e^{-2s} - (d+3)e^{-3s}}{(1-e^{-s})^2}, \tag{3.14}$$

with solution

$$J(t) = \exp\left[2\int_t^{\infty} \frac{e^{-u}}{1-e^{-u}}\, du + dt\right] \int_t^{\infty} \exp\left[-2\int_s^{\infty} \frac{e^{-u}}{1-e^{-u}}\, du - ds\right] \left[\frac{(d+4)e^{-2s} - (d+3)e^{-3s}}{(1-e^{-s})^2}\right] ds.$$

$$\tag{3.16}$$

and for $0 < t \ll 1$,

$$J(t) \simeq \frac{k}{t^2} + \frac{k(2+d) - 1}{t} \tag{3.16}$$

where

$$k = \int_0^{\infty} \exp\left[-2\int_s^{\infty} \frac{e^{-u}}{1-e^{-u}}\, du - ds + 2\gamma\right] \left[\frac{(d+4)e^{-2s} - (d+3)e^{-3s}}{(1-e^{-s})^2}\right] ds \tag{3.17}$$

with

$$\gamma = \int_0^1 \left(\frac{e^{-u}}{(1-e^{-u})} - \frac{1}{u}\right) du + \int_1^{\infty} \frac{e^{-u}}{(1-e^{-u})}\, du. \tag{3.18}$$

The linear solutions to (3.5), (3.6) are given below.

Setting

$$m(a) \equiv \alpha_1 a + \beta_1 \tag{3.19}$$

$$r(a) = \alpha_2 a + \beta_2$$

in (3.5), (3.6) respectively, and equating coefficients of a^2, a, 1, respectively, after clearing the denominators, one obtains for the Renyi model

$$a(\alpha_1 a+\beta_1) = a + 2\left[\sum_{\ell=d+3}^{a-d-1} (\alpha_1 \ell + \beta_1)\right] \tag{3.20)(i}$$

$$+ 2(d+2+\frac{2}{d+2}),$$

since

$$m(d+2) = 1 + \frac{2}{d+2} , \tag{3.20)(ii}$$

and one obtains

$$\alpha_1 = \frac{3d^2+11d+12}{2(d+2)(2d^2+5d+3)} , \beta_1 = \alpha_1(2d+1)-1, \tag{3.21)(i}$$

and for the Solomon model, the corresponding result is

$$\alpha_2 = \frac{2}{d+2} , \beta_2 = \frac{d}{d+2} . \tag{3.21)(ii}$$

The variances for the two models are obtained by squaring (1.24) to obtain, denoting

$$m_2(a) \equiv E(X(a))^2 \tag{3.22)(i}$$

$$r_2(a) \equiv E(Y(a))^2, \tag{3.22)(ii}$$

that, respectively,

$$m_2(a) = 2m(a) - 1 + \frac{2}{a}\sum_{\ell=1}^{a-2d-2} m(\ell)m(a-2d-\ell-1) + \frac{2}{a}\sum_{\ell=1}^{a-d-1} m_2(\ell) \qquad (3.23)$$

with $m_2(\ell) = 1,\ 1 \leq \ell \leq d,\ m_2(d+1) = 1 + \frac{6}{d}$

$$r_2(a) = 2r(a) - 1 + \frac{2}{a+d}\sum_{\ell=1}^{a-d-2} r(\ell)r(a-d-\ell-1) + \frac{2}{a+d}\sum_{\ell=1}^{a-1} r_2(\ell) \qquad (3.24)$$

with

$$r_2(1) = 1,\ r_2(2) = 1 + \frac{2}{d+2}\ .$$

Denoting the linear functions, for $x \geq d$

$$\ell_1(x) = bx + (2b(d+1)-1) \simeq m(x) \qquad (3.25)(i)$$

$$\ell_2(x) = kx + (k(d+2)-1) \simeq r(x) \qquad (3.25)(ii)$$

corresponding to the means for the Renyi and Solomon cases, respectively.

$$(i)\quad s^2(a) = E(X(a)-\ell_1(a))^2 \qquad (3.26)(i)$$

$$(ii)\quad t^2(a) = E(Y(a)-\ell_2(a))^2. \qquad (3.26)(ii)$$

From (3.23), (3.24) respectively, it follows that

$$s^2(a) = f_1(a) + \frac{2}{a}\sum_{v=1}^{a-d-1} (m(v)-\ell_1(v))(m(a-2d-v-1)-\ell_1(a-2d-v-1))$$

$$+ \frac{2}{a}\sum_{v=1}^{a-d-1} s^2(v) \qquad (3.27)$$

$$t^2(a) = f_2(a) + \frac{2}{a+d} \sum_{v=1}^{a-d-2} (r(v)-\ell_2(v))(r(a-d-v-1)-\ell_2(a-d-v-1))$$

$$+ \frac{2}{a+d} \sum_{v=1}^{a-1} t^2(v), \qquad (3.28)$$

where $|f_i(a)| \leq M$, $i=1,2$, for some constant $M < \infty$.

From the Laplace transform of (3.27), (3.28), it may be concluded that

$$a^{-1}s^2(a) \to C_1 < \infty \qquad (3.29)(i)$$

$$a^{-1}t^2(a) \to C_2 < \infty \qquad (3.29)(ii)$$

and hence that (3.4)(iii), (3.4)(iv) hold. This completes the theorem.

IV. Random Car Size

One covering model by randomly chosen car lengths is as follows. On a curb [0,a] a car of length x is placed with center chosen uniformly on [0,a], and x is chosen from a probability distribution with density f(x). The car is kept in place (≡parked) if and only if its center is on [0,a]. A second car is chosen I.I.D. as the first, and parked if and only if its center is on an uncovered portion of [0,a]. The process continues until no further portion of the curb is uncovered.

Let

$$F(x) = \int_c^x f(y)dy, \text{ with } F(c)=0,\ F(c+) > 0, \text{ for some } c > 0 \qquad (4.1)$$

denote the distribution function of one-half of the car length, and

M(a) = mean total number of cars required to first cover [0,a]. (4.2)

Then by considerations as in (1.4) for $a > c$,

$$M(a) = 1 + \frac{2}{a}\int_0^a f(u)du \int_0^{a-u} M(y)dy \tag{4.3}$$

and $M(x) = 1$, $0 < x < c$.

A formal solution for the Laplace transform of may now be obtained.

It is assumed that $M(a)$ is finite. This can be guaranteed, e.g., by setting $F(c) = 0$ for some $c > 0$.

Assume that

$$\int_0^\infty y^2 f(y)dy < \infty. \tag{4.4}$$

Denote

$$\Phi(s) = \int_0^\infty e^{-sy} f(y)dy. \tag{4.5}$$

$$L(s) = \int_c^\infty e^{-sy} M(y)dy. \tag{4.6}$$

Then multiplying (4.3) by e^{-sa} and integrating from 0 to ∞ yields

$$L'(s) + \frac{2\Phi(s)}{s} L(s) = -\frac{e^{-sc}(1+sc) + 2\Phi(s)(1-e^{-sc})}{s^2} \tag{4.7}$$

with solution

$$L(s) = \exp\left[2\int_s^\infty \frac{\Phi(u)}{u}\,du\right] \int_s^\infty \exp\left[-2\int_v^\infty \frac{\Phi(u)}{u}\,du + 2\delta\right] \left[\frac{e^{-vc}(1+vc)+2\Phi(v)(1-e^{-vc})}{v^2}\right] dv \tag{4.8}$$

with

$$\delta = \int_0^1 \frac{\Phi(u)-1}{u}\,du + \int_1^\infty \frac{\Phi(u)}{u}\,du. \tag{4.9}$$

For $0 < s \ll 1$,

$$L(s) \simeq \frac{\gamma}{s^2} + \frac{2\mu\gamma-1}{s} \tag{4.10}$$

where here

$$\gamma = \int_0^\infty \exp\left[-2\int_v^\infty \frac{\Phi(u)}{u}\,du + 2\delta\right]\left[\frac{e^{-vc}(1+vc)+2\Phi(v)(1-e^{-vc})}{v^2}\right] dv \tag{4.11}$$

and

$$\mu = \int_0^\infty y f(y)\,dy. \tag{4.12}$$

Theorem IV. Under the hypotheses of section IV,

$$\lim_{a\to\infty} a^{-1}M(a) = \gamma. \tag{4.13}$$

Proof. This follows from (4.10) and a Tauberian theorem (Widder 1941), pp. 182, 192), since M(a) is of bounded variation.

Remark I. A simple approximation to γ may be obtained by finding the approximate linear solution to (4.3). Assuming a solution to (4.3) of form $\alpha a+\beta$, one equates coefficients of $a^2, a, 1$ for a large, in

$$a(\alpha a+\beta) = a + 2\int_0^a f(u)\,du\left[\int_c^{a-u} (\alpha y+\beta)\,dy + c\right], \tag{4.14}$$

that is, solve for α,β in

$$\alpha a^2 + \beta a \cong a + \alpha[a^2 - 2a\mu + \mu_2 - c^2] + 2\beta[a-u-c] + 2c \tag{4.15}$$

where

$$\mu_2 = \int_c^\infty y^2 f(y)dy, \tag{4.16}$$

to obtain

$$\alpha = \frac{2\mu + 4c}{c^2 + 4\mu^2 + 4\mu c - \mu_2}, \quad \beta = 2\alpha\mu - 1 \tag{4.17}$$

The properties of this approximation remain to be investigated. It is conjectured to be a good upper bound. When $c = \mu = \frac{1}{2}$, $\mu_2 = \frac{1}{4}$, corresponding to unit length cars, $\alpha = \frac{3}{2}$, which is a good upper bound in the Renyi covering model of section I.

Remark II. Let

$W(a)$ = total number of random size cars required to cover $[0,a]$. (4.18)

Let

$$M_2(a) = EW^2(a) \tag{4.19}$$

By the usual argument as in (1.4),

$$M_2(a) = 2M(a) - 1 + \frac{2}{a}\int_0^a f(u)du \int_0^{a-u} M(y)M(a-u-y)dy$$

$$+ \frac{2}{a}\int_0^a f(u)du \int_0^{a-u} M_2(y)dy \tag{4.20}$$

and $M_2(y) = 1$, $0 \le y < c$.

Since this expression is similar in form to the second moment expressions for the models of previous sections, it is plausible that

$$\mathrm{Var}[W(a)] = M_2(a)-(M(a))^2 \simeq E(W(a)-\ell(a))^2 \tag{4.21}$$

where

$$\ell(a) = \gamma a + 2\mu\gamma - 1, \tag{4.22}$$

and hence it is conjectured that, under mild moment conditions,

$$\lim_{a\to\infty} a^{-1}\mathrm{Var}[W(a)] = \eta > 0, \tag{4.23}$$

where η is a finite constant.

V. Alternating car size

On a curb $[0,x]$, a car of length 1 is parked in accord with a Renyi model (i.e. the center must be on $[0,x]$) or a Solomon model (i.e. some portion must be on $[0,x]$). The next car candidate in the same model is of length $a > 1$, and in the Renyi model is parked if and only if, the center falls on an uncovered portion of $[0,x]$, and, for $a > 2$, the car does not completely contain the car of length 1. In the Solomon model, some portion of the a-length car must fall on an uncovered portion of $[0,x]$, and also must not completely contain a car of length 1, otherwise it is discarded. If an a-length car is discarded in either model, another a-length car is selected I.I.D. as the prior one, until one is parked, or the interval $[0,x]$ is completely covered, at which time the process stops. If the a-length car is parked, and some portion of [0,x] is uncovered in either model, the next car to be parked is one of length 1, followed by one of length a, and so on, alternately, until [0,x] is completely covered in either model, at which time the process stops.

Let

$X_1(x)$ = minimum total number of alternate size cars required to cover [0,x] in the Renyi model starting with a car of length 1, and $X_2(x)$ = same quantity starting with a car of length a.

$X^{(1)}(u)$ = minimum total number of alternate size cars required to cover [0,u] in the Renyi model when the first car to be parked was an a-length car, parked at [u,u+a] on a total interval [0,x], for $x > u+a$.

$X^{(2)}(u)$ = minimum total number of alternate size cars required to cover [0,u] in the Renyi model when the first car to be parked was a 1-length car, parked at [u,u+1] on a total interval [0,x], for $x > u+1$.

Let (5.2)

$Y_1(x)$, $Y_2(x)$, $Y^{(1)}(u)$, $Y^{(2)}(u)$ be the corresponding quantities for the Solomon model.

Let (5.3)

$$r_i(x) = E(Y_i(x)) \qquad i=1,2$$
$$r^{(i)}(x) = E(Y^{(i)}(x)) \quad i=1,2.$$

By considerations as in the previous sections, by expressions similar to (1.4), it follows, for the Solomon model, for example,

$$r_2(x) = \frac{2}{x+a}\int_{-a}^{0} 1 + r_1(x-u-a)du$$

$$+\frac{2}{x+a}\int_0^{x-a} du\left[1+\left(\frac{u+1}{x-a+2}\right)(r_1(u)+r^{(2)}(x-u-a))\right]. \qquad (5.4)$$

$$r_1(x) = \frac{2}{x+1}\int_{-1}^{0} du(1+r_2(x+1-2a-u)$$

$$+\frac{2}{x+1}\int_0^{x-1} du\left[1+\left(\frac{u+2-2a}{x+3-4a}\right)\left(r_2(u+2-2a)+r^{(1)}(x-u-1)\right),\right.$$

with

$$r_1(0) = r_2(0) = 1. \qquad (5.4)(ii)$$

Also, an approximation which may be good for large x is

$$\int_0^{x-a} du\ r^{(1)}(u) = \int_0^{x-a} du\left(\frac{u+1}{x-a+2}\right) r_1(u)$$

$$+\int_0^{x-a} du\left(\frac{x-a+1-u}{x-a+2}\right) r^{(2)}(u). \qquad (5.4)(iii)$$

Similar expressions hold for the means in the Renyi models, with slightly different initial conditions, and the cases $1 < a < 2$ and $a > 2$ are treated separately. It may be possible to obtain linear solutions to (5.4) for x large, but the computation is tedious, and of unknown merit. The exact solution of (5.4) by Laplace transforms appears possible by power series or a Volterra-like expansion, and an asymptotic solution with leading term s^{-2} is sought, corresponding to asymptotic linearity, since it is conjectured that $r_i(x)$, $i=1,2$, grow, for $x \to \infty$ as

$$r_i(x) \sim \alpha_i x + \beta_i,\ i=1,2. \qquad (5.5)$$

This is because $r_a(x) \leq r_1(x) \leq r(x)$, where $r(x)$ is the mean number of unit length cars required to cover $[0,x]$, and $r_a(x)$ is the mean number of a-length cars required to cover $[0,x]$ in the Solomon model.

VI. Abacus Packing.

A one-dimensional abacus packing (≡ parking) model on a discrete grid is considered. A car of length d, where d is an odd, positive integer, is parked horizontally with center chosen uniformly from $\left(\frac{d+1}{2},0\right), \left(\frac{d+3}{2},0\right),\ldots\left(a-\left(\frac{d-1}{2}\right),0\right)$ where a is a positive integer, $a \geq d$. A second car of length d is chosen I.I.D. as the first, and parked if and only if it does not intersect the first car, and its end closest to the first car is at least 1 unit away, otherwise it is discarded, and another car chosen I.I.D. as the first, and so on, until it is parked or no further cars may fit in the prescribed manner.

Let (6.1)

$X(a)$ = total number of d-length cars (d an odd, positive integer) which may be parked on $\{(\ell,0)\ 1 \leq \ell \leq a\}$, with center at $(k,0)$, with k chosen uniformly from: $\frac{d+1}{2} \leq k \leq a - \left(\frac{d-1}{2}\right)$.

Then, conditional on the center k chosen as above,

$$X(a) = \begin{cases} 1 + X'\left(k-\left(\frac{d+3}{2}\right)\right) + X''\left(a-k-\left(\frac{d+1}{2}\right)\right) & \text{if } \frac{d+3}{2} \leq k \leq a - \left(\frac{d+1}{2}\right) \\ 1 + X'(a-d-1) & \text{if } k = \frac{d+1}{2} \text{ or } k = a-\left(\frac{d-1}{2}\right) \end{cases} \tag{6.2}$$

where $X'(r)$, $X''(r)$ are I.I.D. as $X(r)$.

Let, for $r \geq 0$ an integer,

$$m(r) \equiv EX(r) \tag{6.3}$$

Then, from (6.2), on taking expectations, one may obtain

$$m(a+d-1) = 1 + \frac{2}{a}\sum_{\ell=1}^{a-2} m(\ell) \tag{6.4}$$

with $m(\ell) = 1$, $d \le \ell \le 2d$, $m(\ell) = 0$, $\ell < d$.

To obtain the asymptotic mean number of d-length cars which may be parked, one proceeds as before using the Laplace transform.

Define

$$K(s) \equiv \sum_{\ell=d}^{\infty} m(\ell)e^{-s\ell} \tag{6.5}$$

Then (6.4) yields that, summing from $\ell=1$ to ∞,

$$K'(s) + \left(d-1 + \frac{2e^{-s(d+1)}}{1-e^{-s}}\right)K(s) = \frac{-e^{-sd}}{(1-e^{-s})^2} \tag{6.6}$$

with solution

$$K(s)=\exp\left[-(d-1)s+2\int_s^\infty \frac{e^{-u(d+1)}}{(1-e^{-u})}\,du\right]\int_s^\infty \exp\left[(d-1)v-2\int_v^\infty \frac{e^{-u(d+1)}}{1-e^{-u}}\,du\right]\left[\frac{e^{-dv}}{(1-e^{-v})^2}\right]dv, \tag{6.7}$$

so that for $0 < s \ll 1$,

$$K(s) \simeq \frac{c}{s^2} + \frac{c(d+3)-1}{s} \tag{6.8}$$

where

$$c = \int_0^\infty \exp\left[(d-1)v-2\int_v^\infty \frac{e^{-u(d+1)}}{(1-e^{-u})}\,du + 2\delta\right]\left[\frac{e^{-dv}}{(1-e^{-v})^2}\right]dv \tag{6.9}$$

with

$$\delta = \int_1^{\infty} \frac{e^{-u(d+1)}}{(1-e^{-u})} du + \int_0^1 \left[\frac{e^{-u(d+1)}}{(1-e^{-u})} - \frac{1}{u} \right] du. \tag{6.10}$$

A Tauberian argument (Widder (1941), pp. 182, 192), yields, for ℓ large

$$m(\ell) \sim c\ell + c(d+3) - 1. \tag{6.11}$$

<u>Remark 1</u>. An approximation to $m(\ell)$ in (6.11) may be obtained by obtaining the linear solution to (6.4), as follows.

Writing (6.4), for $a \geq 2d + 3$ as

$$a\, m(a+d-1) = a + 2 \left[\sum_{\ell=2d+1}^{a-2} m(\ell) \right] + 2(d+1) \tag{6.12}$$

and trying a solution of the form

$$r(\ell) = \alpha\ell + \beta \tag{6.13}$$

and equating coefficients of a^2, a, 1 respectively yields the solution

$$r(\ell) = \frac{3\ell + 3 - d}{4d + 3} \quad . \tag{6.14}$$

<u>Remark 2</u>. The properties of a linear solution depend on d. The case d = 1 is given, as in Weiner (1980).

The equation (6.4) becomes

$$m(a) = 1 + \frac{2}{a} \sum_{\ell=1}^{a-2} m(\ell) \tag{6.15}$$

with $m(1) = m(2) = 1$.

<u>Lemma 2</u>. If for $\ell \geq r + 2$, for some $r \geq 1$,

$$\Delta(\ell) = \frac{2}{\ell} \sum_{k=r}^{\ell-2} \Delta(k) \tag{6.16}$$

and $\Delta(\ell_0+1) \geq 0$, $\Delta(\ell_0+2) \geq 0$, $\ell_0+1 \geq r + 2$, then $\Delta(\ell) \geq 0$, $\ell \geq \ell_0 + 1$.

<u>Proof</u>.

$$\Delta(\ell_0+2) \geq 0 \text{ implies } \sum_{k=r}^{\ell_0} \Delta(k) \geq 0, \tag{6.17}$$

so that

$$\Delta(\ell_0+3) = \frac{2}{\ell_0+3}\left[\sum_{k=r}^{\ell_0} \Delta(k) + \Delta(\ell_0+1)\right] \geq 0. \tag{6.18}$$

Now $\Delta(\ell_0 + 2) \geq 0$, $\Delta(\ell_0+3) \geq 0$, and a repeat of the argument yields a proof by induction.

<u>Lemma 3</u>. For $\ell \geq 5$,

$$\frac{3\ell+2}{7} \leq m(\ell) \leq \frac{4\ell+3}{9} \tag{6.19}$$

<u>Proof</u>. Eq. (6.15) may be written

$$m(\ell) = 1 + \frac{2}{\ell} + \frac{2}{\ell} \sum_{k=2}^{\ell-2} m(k). \tag{6.20}$$

For $\ell \geq 4$, it is immediately verified that the function

$$r(\ell) \equiv \frac{4\ell+3}{9} \tag{6.21}$$

satisfies

$$r(\ell) = 1 + \frac{2}{\ell} + \frac{2}{\ell} \sum_{k=2}^{\ell-2} r(k). \tag{6.22}$$

Denoting

$$\Delta_1(\ell) = r(\ell) - m(\ell), \tag{6.23}$$

then

$$\Delta_1(\ell) = \frac{2}{\ell} \sum_{k=2}^{\ell-2} \Delta_1(k) \tag{6.24)(i}$$

and a computation yields that

$$\Delta_1(5) \geq 0, \ \Delta_1(6) \geq 0. \tag{6.24)(ii}$$

Hence lemma 1 applied to (6.22)-(6.24) yields the right side inequality of (6.19).

For $\ell \geq 5$, (6.15) may be written

$$m(\ell) = 1 + \frac{4}{\ell} + \frac{2}{\ell} \sum_{k=3}^{\ell-2} m(k) \tag{6.25}$$

and for $\ell \geq 5$, the function

$$V(\ell) \equiv \frac{3\ell + 2}{7} \tag{6.26}$$

satisfies

$$V(\ell) = 1 + \frac{4}{\ell} + \frac{2}{\ell} \sum_{k=3}^{\ell-2} V(k). \tag{6.27}$$

Denoting

$$\Delta_2(\ell) = m(\ell) - V(\ell), \tag{6.28}$$

then

$$\Delta_2(\ell) = \frac{2}{\ell} \sum_{k=3}^{\ell-2} \Delta_2(k) \tag{6.29}$$

and a computation yields that

$$\Delta_2(5) \geq 0, \ \Delta_2(6) \geq 0, \tag{6.30}$$

so that lemma 1 applied to (6.28)-(6.30) yields the left inequality of (6.19).

By arguments for the second moments as in the previous sections, it may be seen that the

$$\lim_{x \to \infty} x^{-1} \mathrm{Var}\, X(x) = \gamma > 0 \tag{6.31}$$

VII. Remarks.

Remark 1. Using Weiner (1979), it is easily seen from higher asymptotic moment computations, obtained by taking expectations of higher powers of (1.24) that a central limit theorem holds for each model in sections I, II. One example is the following.

Let (7.1)

$X(x)$ = total number of cars required to cover $[0,x]$ in either a Renyi or Solomon model of sections I, II.

Denote

$$\lim_{x \to \infty} x^{-1} E(X(x)) \equiv \mu > 0 \qquad (7.2)(i)$$

$$\lim_{x \to \infty} x^{-1} \mathrm{Var}(X(x)) = \sigma^2 > 0. \tag{7.2)(ii}$$

Theorem V. For $a < b$,

$$\lim_{x \to \infty} P\left[a < \frac{X(x) - \mu x}{\sigma \sqrt{x}} < b\right] = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-y^2/2} dy. \tag{7.3}$$

Remark 2.

The same considerations are conjectured to suffice to establish a central limit theorem for models in III, IV, VI, and a bivariate central limit theorem for the alternate car size model of section V.

Remark 3. It is conjectured that a two dimensional "Palasti" result holds for the covering models of sections I, II. That is, for unit square cars dropped uniformly on a rectangular parking area of size $x \times y$, let (7.4)

$M(x,y)$ = mean total number of unit square cars required to cover $x \times y$ under a Renyi or Solomon model.

The Palasti conjecture for this model is then

$$\lim_{x,y \to \infty} (xy)^{-1} M(x,y) = \mu^2. \tag{7.5}$$

See Weiner (1978) for a heuristic discussion of packing models in the plane.

REFERENCES

Blaisdell, E., and Solomon, H. (1970): On random sequential packing in the plane and a conjecture of Palasti. Jour. Applied Prob. **7**, 667-698.

Solomon, H. (1966): Random packing density. Proc. Fifth Berkeley Symposium on Math. Stat. and Prob., U. California Press, 119-134.

Weiner, H. (1978): Sequential random packing in the plane, Jour. Applied Prob., **15**, 803-814.

__________ (1979): Central limit theorem for random packing on the line and plane, Sankhyā, A (to appear).

__________ (1980): Constrained random packing in the plane. Tech. Report Stanford University (to appear).

Widder, D. V. (1941), The Laplace Transform, Princeton University Press, Princeton, N.J.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 314	2. GOVT ACCESSION NO. AD-A115 392	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) RANDOM COVERING AND PACKING ON THE LINE		5. TYPE OF REPORT & PERIOD COVERED TECHNICAL REPORT
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) HOWARD J. WEINER		8. CONTRACT OR GRANT NUMBER(s) N00014-76-C-0475
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Statistics Stanford University Stanford, CA 94305		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR-042-267
11. CONTROLLING OFFICE NAME AND ADDRESS Office Of Naval Research Statistics & Probability Program Code 436 Arlington, VA 22217		12. REPORT DATE FEBRUARY 16, 1982
		13. NUMBER OF PAGES 31
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Asymptotic random covering; integral equations; packing; asymptotic approximations.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

PLEASE SEE REVERSE SIDE.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-LF-014-6601

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

#314

RANDOM COVERING AND PACKING ON THE LINE

The asymptotic expected mean of the minimum number of randomly, uniformly placed unit intervals needed to cover [0,x) is obtained for the models of Renyi and Solomon. Asymptotic variances are indicated, and other models are given. The mean number of unit length intervals which may pack or cover a one-dimensional abacus grid is given.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)